Comment. Math.
Univ. St. Pauli
XIX—1, 1970

# Independence proof of $(\omega, \omega_\alpha)$-distributive law in complete Boolean algebras[1)]

by

Kanji NAMBA

(Received March 16, 1970)

In this paper, we answer the following problem proposed by D. Scott and R. Solovay in [2] concerning the $(\omega, \omega_\alpha)$-distributive law (abbreviated as $(\omega, \omega_\alpha)$-DL) for complete Boolean algebras: *For what kind of cardinal number $\omega_\alpha$ is there a complete Boolean algebra $\boldsymbol{B}$ such that*

(1) $\forall\nu<\alpha[(\omega, \omega_\nu)$*-DL holds in* $\boldsymbol{B}]$,
(2) $(\omega, \omega_\alpha)$*-DL fails to hold in* $\boldsymbol{B}$?

The conditions (1) and (2) clearly imply that

(a) $\forall\nu<\alpha[\overline{\overline{\omega_\nu^\omega}}<\omega_\alpha]$.

On the other hand, as is shown by Lemma 2 below, if there is a complete Boolean algebra satisfying (1) and (2), then

(b) $cf(\omega_\alpha)=\omega$

or

(b′) $\omega_\alpha$ is regular.

For the case that (a) and (b) hold, the existence of such a Boolean algebra has been shown by K. Prikry [3]. Here we shall show that the situation is the same even for the remaining case where (b) is replaced by (b′) (Theorem 3 below).

First, to compare our result, we refer to Prikry's theorem:

THEOREM 1 (Karel Prikry). *Let $\omega_\alpha$ satisfy the conditions*

$$cf(\omega_\alpha)=\omega, \qquad \forall\nu<\alpha[\overline{\overline{\omega_\nu^\omega}}<\omega_\alpha] .$$

*Then there is a complete Boolean algebra $\boldsymbol{B}$ satisfying*
(1) $\forall\nu<\alpha[(\omega, \omega_\nu)$*-DL holds in* $\boldsymbol{B}]$,
(2) $(\omega, \omega_\alpha)$*-DL fails to hold in* $\boldsymbol{B}$.

We do not carry out here the proof of this theorem.

LEMMA 2. *Let $\omega_\alpha$ satisfy the condition $\omega<cf(\omega_\alpha)<\omega_\alpha$. Then*

$$\forall\nu<\alpha[(\omega, \omega_\nu)\text{-}DL \text{ holds in } \boldsymbol{B}]$$

*implies that*

$$(\omega, \omega_\alpha)\text{-}DL \text{ holds in } \boldsymbol{B} .$$

1) This work was partially supported by Matsunaga Science Foundation.

*Proof*. Assume that $\boldsymbol{b}=\prod_{n<\omega}\sum_{\nu<\omega_\alpha}\boldsymbol{a}_{n\nu}>\boldsymbol{0}$. Let $\omega_\beta=cf(\omega_\alpha)<\omega_\alpha$. Then we have a function $f\colon \omega_\beta\to\omega_\alpha$ such that $\sup_{\lambda<\omega_\beta} f(\lambda)=\omega_\alpha$. Hence

$$\sum_{\nu<\omega_\alpha}\boldsymbol{a}_{n\nu}=\sum_{\lambda<\omega_\beta}\sum_{\nu<f(\lambda)}\boldsymbol{a}_{n\nu} .$$

Let $\boldsymbol{b}_{n\lambda}=\sum_{\nu<f(\lambda)}\boldsymbol{a}_{n\nu}$. Then we obtain that

$$\boldsymbol{b}=\prod_{n<\omega}\sum_{\lambda<\omega_\beta}\boldsymbol{b}_{n\lambda} .$$

By $(\omega, \omega_\beta)$-DL, we have

$$\boldsymbol{b}=\sum_{g\in\omega_\beta^\omega}\prod_{n<\omega}\boldsymbol{b}_{n\,g(n)} .$$

Since $g\in\omega_\beta^\omega$ and $cf(\omega_\alpha)>\omega$, we have $\sup_{n<\omega} f(g(n))<\omega_\alpha$. From the fact that $\forall\nu<\alpha[(\omega, \omega_\nu)$-DL holds in $\mathbf{B}]$, we have

$$\prod_{n<\omega}\boldsymbol{b}_{n\,g(n)}=\sum_{t\in\prod_{m<\omega} f(g(m))}\prod_{n<\omega}\boldsymbol{a}_{n\,t(n)} .$$

Therefore we have

$$\boldsymbol{b}=\sum_{g\in\omega_\beta^\omega}\sum_{t\in\prod_{m<\omega} f(g(m))}\prod_{n<\omega}\boldsymbol{a}_{n\,t(n)}=\sum_{t<\omega_\alpha^\omega}\prod_{n<\omega}\boldsymbol{a}_{n\,t(n)} .$$

Namely $(\omega, \omega_\alpha)$-DL holds in $\boldsymbol{B}$.

THEOREM 3. *Let $\omega_\alpha$ satisfy the following conditions*:

$$\forall\nu<\alpha[\overline{\overline{\omega_\nu^\omega}}<\omega_\alpha], \quad \omega_\alpha \textit{ is regular.}$$

*Then there is a complete Boolean algebra $\boldsymbol{B}$ satisfying*

(1) $\forall\nu<\alpha[(\omega, \omega_\nu)$*-DL holds in* $\boldsymbol{B}]$,

(2) $(\omega, \omega_\alpha)$*-DL fails to hold in* $\boldsymbol{B}$.

To prove this theorem, we need some preliminaries.

Let $\omega_\alpha$ be a regular cardinal such that $\forall\nu<\alpha[\overline{\overline{\omega_\nu^\omega}}<\omega_\alpha]$, and let

$$A\subset\omega_\alpha^\omega$$

($A$ is a set of functions on $\omega$ to $\omega_\alpha$). By $A^*$ we denote the set

$$A^*=\{f\restriction n : n<\omega \wedge f\in A\} .$$

If for any increasing functions $g_0\subset g_1\subset\cdots\subset g_n\subset\cdots$ of $A^*$,

$$\bigcup_{n<\omega} g_n\in\omega_\alpha^\omega\to\bigcup_{n<\omega} g_n\in A ,$$

then $A$ is called a *closed* set. If $A$ satisfies the following two conditions:

(1) $A$ is closed,

(2) $\forall f\in A^*[\overline{\overline{\{\nu\colon f*\nu\in A^*\}}}=\omega_\alpha]$,

where $f*\nu$ is the concatenation of $f$ and $\nu$, then $A$ is called $\omega_\alpha^*$*-perfect*.

Let $\boldsymbol{Q}$ be the set of all $\omega_\alpha^*$-perfect sets. For $f \in (\omega_\alpha^\omega)^*$ and $A \subset \omega_\alpha^\omega$, define $f*A$ by

$$f*A = \{f*g : g \in A\} .$$

Put

$$\boldsymbol{P} = \{f*A : f \in (\omega_\alpha^\omega)^* \wedge A \in \boldsymbol{Q}\} ,$$

and define an order relation $\leqq$ on $\boldsymbol{P}$ by

$$f_1*A_1 \leqq f_2*A_2 \equiv f_1*A_1 \supset f_2*A_2 .$$

Then the structure $(\boldsymbol{P}, \leqq)$ becomes a quasi-ordered structure, and taking

$$V(p) = \{q : p \leqq q\}$$

as the unique neighbourhood of $p$, $(\boldsymbol{P}, V)$ becomes a topological space (generally it is not a Hausdorff space). Consider the complete Boolean algebra formed by the regular open sets of this topological space. Let this algebra be $\boldsymbol{B}^*[\omega_\alpha]$ or $\boldsymbol{B}^*$.

Now we shall show that $V(p)$ is regular open in $(\boldsymbol{P}, V)$. In order to show this, it is enough to show that

$$\neg\, q \in V(p) \longrightarrow \exists r \geqq q\ \forall s \geqq r [\neg\, s \in V(p)] .$$

Namely it is enough to show that

$$\neg\, p \leqq q \longrightarrow \exists r \geqq q\ \forall s \geqq r [\neg\, p \leqq s] .$$

Suppose that $\neg\, f_1*A_1 \subset f_0*A_0$. If

$$\forall f \in (\omega_\alpha^\omega)^* [f \in (f_1*A_1)^* \longrightarrow f \in (f_0*A_0)^*] ,$$

then, since $f_1*A_1$ and $f_0*A_0$ are closed, we must have $f_1*A_1 \subset f_0*A_0$. This is a contradiction. Namely there is a function $f_2 \in (\omega_\alpha^\omega)^*$ such that

$$f_2 \in (f_1*A_1)^*, \quad \neg\, f_2 \in (f_0*A_0)^*, \quad \neg\, f_1 \subset f_2 .$$

We can take a set $A_2 \in \boldsymbol{Q}$ so that

$$f_2*A_2 \subset f_1*A_1 .$$

If $f_3*A_3 \subset f_2*A_2$, then $f_2 \subset f_3$. Hence, if $f_3*A_3 \subset f_0*A_0$, then $f_2 \in (f_3*A_3)^* \subset (f_0*A_0)^*$, a contradiction. From this we have

$$\neg\, f_3*A_3 \subset f_0*A_0 .$$

This means that $V(f_0*A_0)$ is a regular open set for all $f_0*A_0 \in \boldsymbol{P}$.

LEMMA 4. *$(\omega, \omega_\alpha)$-DL does not hold in $\boldsymbol{B}^*[\omega_\alpha]$.*

*Proof.* Let $n < \omega$ and $\nu \in \omega_\alpha$. Define

$$\boldsymbol{a}_{n\nu} = \sum_{\substack{\mathrm{dom}(f) = n+1 \\ f(n) = \nu}} V(f*\omega_\alpha^\omega) .$$

By this definition we have

$$1=\sum_{\nu<\omega_\alpha} \boldsymbol{a}_{n\nu}$$

for every $n<\omega$. Hence we obtain

$$1=\prod_{n<\omega}\sum_{\nu<\omega_\alpha} \boldsymbol{a}_{n\nu} .$$

Now assume $g*B\in \boldsymbol{a}_{n\nu}$. Consider the case where $n\in \operatorname{dom}(g)$ and $g(n)\neq\nu$. If $f(n)=\nu$, then

$$V(g*B)\cap V(f*\omega_\alpha^\omega)=\phi .$$

By this we obtain $V(g*B)\cap \boldsymbol{a}_{n\nu}=\phi$. Namely $g*B\notin \boldsymbol{a}_{n\nu}$.

Next consider the case where $n\notin \operatorname{dom}(g)$. There is a $h\in(\omega_\alpha^\omega)^*$ such that

$$h(n)\neq\nu,\quad h*C\subset g*B .$$

This means that $g*B\notin \boldsymbol{a}_{n\nu}$. By these facts, we have that if $g*B\notin \boldsymbol{a}_{n\nu}$, then $n\in\operatorname{dom}(g)$ and $g(n)=\nu$. Assume that

$$\prod_{n<\omega} \boldsymbol{a}_{n\,f(n)}>0$$

for some $f\in\omega_\alpha^\omega$. Then there is $g*B\in \boldsymbol{P}$ such that

$$g*B\in\prod_{n<\omega}\boldsymbol{a}_{n\,f(n)}=\bigcap_{n<\omega}\boldsymbol{a}_{n\,f(n)} .$$

By the above remark, we have $f\subset g$. But $g\in(\omega_\alpha^\omega)^*$, hence $\operatorname{dom}(g)<\omega$, this is a contradiction. Namely we have

$$\sum_{f\in\omega_\alpha^\omega}\prod_{n<\omega}\boldsymbol{a}_{n\,f(n)}=0 .$$

LEMMA 5. *Let $\omega_\alpha$ be a regular cardinal and $\rho<\omega_\alpha$. Let $\boldsymbol{a}_\nu\in B^*$ $(\nu<\rho)$ be such that*

$$\sum_{\nu\in\rho}\boldsymbol{a}_\nu=1 .$$

*Then for every $f_0*A_0\in \boldsymbol{P}$, we have*

$$\overline{\overline{\{\nu:\exists B\in \boldsymbol{Q}\,\exists\mu<\rho[f*\nu*B\subset f_0*A_0\wedge f*\nu*B\in \boldsymbol{a}_\mu]\}}}=\omega_\alpha$$
$$\rightarrow \exists A'\in \boldsymbol{Q}\,\exists\mu<\rho[f*A'\in \boldsymbol{a}_\mu\wedge f*A'\subset f_0*A_0] .$$

*Proof.* Assume that $\overline{\overline{D}}=\omega_\alpha$, where

$$D=\{\nu:\exists B\in \boldsymbol{Q}\,\exists\mu<\rho[f*\nu*B\subset f_0*A_0\wedge f*\nu*B\in \boldsymbol{a}_\mu\} .$$

Then we can choose $B_\nu\in \boldsymbol{Q}$ for every $\nu\in D$ and a function $f_1\colon D\rightarrow\rho$ so that

$$\{\nu: f*\nu*B_\nu\subset f_0*A_0\wedge f*\nu*B_\nu\in \boldsymbol{a}_{f_1(\nu)}\}\supset D .$$

Since $\rho<\omega_\alpha$ and $\omega_\alpha$ is regular, we have a $\mu_0<\rho$ such that

$$E=\{\nu : f*\nu*B_\nu \subset f_0*A_0 \wedge f*\nu*B_\nu \in \boldsymbol{a}_{\mu_0}\}, \quad \overline{\overline{E}}=\omega_\alpha .$$

Put

$$A'=\bigcup_{\nu\in E} \nu*B_\nu .$$

Then $A' \in \boldsymbol{Q}$. Consider the set $f*A'$. For every $g*C\subset f*A'$, take $g'*C'\subset g*C$ so that $\mathrm{dom}\,(g')>\mathrm{dom}\,(f)$. Then

$$g'=f*\nu*g''$$

for a $\nu \in E$ and a function $g''$. We have

$$(g'*C')_{f*\nu}\subset (f*A')_{f*\nu}=B_\nu .$$

Hence $g'*C'\subset f*\nu*B_\nu \in \boldsymbol{a}_{\mu_0}$. Therefore $g'*C' \in \boldsymbol{a}_{\mu_0}$.

This shows that $V(f*A')\subset \boldsymbol{a}_{\mu_0}$. Since $\boldsymbol{a}_{\mu_0}$ is regular open, we obtain that

$$f*A' \in \boldsymbol{a}_{\mu_0} .$$

LEMMA 6. *Let $\omega_\alpha$ be a regular cardinal and $\sum_{\nu<\rho} \boldsymbol{a}_\nu=1$ for some $\rho<\omega_\alpha$. Then for every $f_0*A_0 \in \boldsymbol{P}$, we have $f_0*A'$ such that*

$$f_0*A'\subset f_0*A_0, \quad f_0*A' \in \boldsymbol{a}_\mu \textit{ for some } \mu<\rho .$$

*Proof.* Assume that there is no $f_0*A'\subset f_0*A_0$ such that $\exists\mu<\rho[f_0*A' \in \boldsymbol{a}_\mu]$. Consider the set

$$D=\{g \in A_0^*: \neg\, \exists C \in \boldsymbol{Q}\, \exists\mu<\rho[f_0*g*C \subset f_0*A_0 \wedge f_0*g*C \in \boldsymbol{a}_\mu]\} .$$

By the assumption, we have $\phi \in D\subset A_0^*$. Take $g \in D$, and assume that

$$\overline{\overline{\{\nu : g*\nu \in D\}}}<\omega_\alpha .$$

Then $\overline{\overline{\{\nu : g*\nu \in A_0^* \wedge g*\nu \notin D\}}}=\omega_\alpha$. Namely we have

$$\overline{\overline{\{\nu : \exists C \in \boldsymbol{Q}\, \exists\mu < \rho[f_0*g*\nu*C \subset f_0*A_0 \wedge f_0*g*\nu*C \in \boldsymbol{a}_\mu]\}}}=\omega_\alpha .$$

By Lemma 5, we obtain that

$$\exists C \in \boldsymbol{Q}\, \exists\mu < \rho[f_0*g*C \subset f_0*A_0 \wedge f_0*g*C \in \boldsymbol{a}_\mu] .$$

Namely $g \notin D$, which is a contradiction. Thus we have

$$\{\nu : g*\nu \in D\}=\omega_\alpha .$$

Let $\bar{D}$ be the closure of $D$. Then since $A_0$ is closed, we have

$$\bar{D}\subset A_0, \quad \bar{D} \in \boldsymbol{Q} .$$

Now consider $f_0*\bar{D}$. Since $\sum_{\nu<\rho}\boldsymbol{a}_\nu=\mathbf{1}$, there is $f_0*h*E$ such that

$$f_0*h*E\subset f_0*\bar{D}\subset f_0*A_0,\quad f_0*h*E\in\boldsymbol{a}_\mu \text{ for some } \mu<\rho\ .$$

Hence we have that $h\in D$, contradicting the definition of $D$. Namely we have

$$f_0*A'\subset f_0*A_0,\quad f_0*A'\in\boldsymbol{a}_\mu \text{ for some } \mu<\rho\ .$$

LEMMA 7. *Let $\omega_\alpha$ be a regular cardinal and $\rho<\omega_\alpha$. Suppose that*

$$\prod_{n<\omega}\sum_{\nu<\rho}\boldsymbol{a}_{n\nu}=\mathbf{1}\ .$$

*Then, for every $f_0*A_0\in\boldsymbol{P}$, we have $f_0*A'\subset f_0*A_0$ such that*

$$\forall g\in A'^*\exists\nu<\rho[f_0*g*B_g\in\boldsymbol{a}_{\mathrm{dom}(g)\,\nu}]$$

*for some $B\in\boldsymbol{Q}$, where $B_g=\{h:g*h\in B\}$.*

*Proof*. We define a sequence of sets $E_n\subset A_0^*$ and, for each $f\in E_n$, a set $A^f\in\boldsymbol{Q}$ and an ordinal $\mu_f<\rho$ by the induction on $n$ as follows:

First, let $E_0=\{\phi\}$ and, by Lemma 6, choose $A^\phi$ and $\mu_\phi$ so that

$$f_0*A^\phi\subset f_0*A_0,\quad f_0*A^\phi\in\boldsymbol{a}_{0\,\mu_\phi}\ .$$

Next, assume that $E_n$ and, for each $f\in E_n$, $A^f$ and $\mu_f$ are already defined, and that

$$f_0*f*A^f\subset f_0*A_0,\quad f_0*f*A^f\in\boldsymbol{a}_{n\,\mu_f}\ .$$

Define $E_{n+1}$ by $f*\nu\in E_{n+1}$ if and only if $f\in E_n$ and $f*\nu\in(A^f)^*$, and, again by Lemma 6, choose $A^{f*\nu}$ and $\mu_{f*\nu}$ so that

$$f_0*f*\nu\in A^{f*\nu}\subset f_0*f*A^f,\quad f_0*f*\nu*A^{f*\nu}\in\boldsymbol{a}_{n+1\,\mu_{f*\nu}}\ .$$

Put $E=\bigcup_{n<\omega}E_n$. Since $E\subset A_0^*$, $E_f\subset A^f$ and

$$\forall f\in E[\overline{\overline{\{\nu:f*\nu\in E\}}}=\omega_\alpha]\ ,$$

it follows that $\bar{E}\in\boldsymbol{Q}$, $\bar{E}\subset A_0$, and $\overline{E_f}=(\bar{E})_f\subset A^f$. Hence $f_0*\bar{E}\subset f_0*A_0$ and $\forall g\in E\,\exists\nu<\rho[f_0*g*\bar{E}_g\in\boldsymbol{a}_{\mathrm{dom}(g)\,\nu}]$.

*Definition*. Let $\boldsymbol{D}_n$ be the set defined by

$$\boldsymbol{D}_n=\{D\subset\omega_\alpha^\omega:\forall g\in D^*[\mathrm{dom}(g)=n\longrightarrow\overline{\overline{\{\nu:g*\nu\in D^*\}}}<\omega_\alpha]\}\ .$$

For $A\subset\omega_\alpha^\omega$, if there is a sequence $D_0, D_1, \cdots, D_n, \cdots$ such that

$$\forall n<\omega[D_n\in\boldsymbol{D}_n],\quad A\subset\bigcup_{n<\omega}D_n\ ,$$

then $A$ is *poor*. Otherwise $A$ is *rich*.

LEMMA 8. *Let $A$ be a rich set, Then*

$$\overline{\overline{\{\nu:A_\nu\ is\ rich\}}}=\omega_\alpha\ ,$$

*where* $A_\nu=\{f: \nu*f \in A\}$.

*Proof.* Assume that $A$ is rich but $\overline{\overline{E}}<\omega_\alpha$, where

$$E=\{\nu: A_\nu \text{ is rich}\} .$$

Then there is a set $D_0 \in \boldsymbol{D}_0$ such that $\bigcup_{\nu \in E} \nu*A_\nu \subset D_0$. By definition, if $\nu \in A^*-E$, then $A_\nu$ is poor. So there is a sequence $D_0^\nu, D_1^\nu, \cdots, D_n^\nu, \cdots$ such that $\forall n<\omega[D_n^\nu \in \boldsymbol{D}_n]$ and

$$A_\nu \subset \bigcup_{n<\omega} D_n^\nu .$$

Now we put $D_{n+1}=\bigcup_{\nu \in A^*-E} \nu*D_n^\nu$. Then $D_{n+1} \in \boldsymbol{D}_{n+1}$. By the definition of $D_{n+1}$, we have

$$\bigcup_{\nu \in A^*-E} \nu*A_\nu \subset \bigcup_{\nu \in A^*-E} \bigcup_{n<\omega} \nu*D_n^\nu = \bigcup_{n<\omega} \bigcup_{\nu \in A^*-E} \nu*D_n^\nu = \bigcup_{n<\omega} D_{n+1} .$$

Hence we obtain

$$A=\bigcup_{\nu \in E} \nu*A_\nu \cup \bigcup_{\nu \in A^*-E} \nu*A_\nu \subset D_0 \cup \bigcup_{n<\omega} D_{n+1} = \bigcup_{n<\omega} D_n .$$

This means that $A$ is poor, contradicting the assumption. Namely we have

$$\overline{\overline{\{\nu: A_\nu \text{ is rich}\}}}=\omega_\alpha .$$

LEMMA 9. *Let $A$ be colsed and rich. Then there is a $B \in \boldsymbol{Q}$ such that*

$$B \subset A .$$

*Proof.* We define sets $C_n \subset A^*$ and $A(f) \subset \omega_\alpha$ for $f \in C_n$ by the induction on $n$. First put $C_0=\{\phi\}$ and $A(\phi)=\{\nu: A_\nu \text{ is rich}\}$. Next assume that $C_n$ is chosen and, for $f \in C_n$, $A(f)$ is already defined so that

$$A(f)=\{\nu: A_{f*\nu} \text{ is rich}\} .$$

Now we put $C_{n+1}=\{f*\nu: f \in C_n \wedge \nu \in A(f)\}$ and, for such $f*\nu$, $A(f*\nu)=\{\mu: A_{f*\nu*\mu} \text{ is rich}\}$. Then, setting $C=\bigcup_{n<\omega} C_n$, we have, by Lemma 8,

$$\phi \in C \subset A^*, \quad \forall f \in C[\overline{\overline{\{\nu: f*\nu*C\}}}=\omega_\alpha] .$$

Since $A$ is closed, we have $\bar{C} \subset A$ and $\bar{C} \in \boldsymbol{Q}$.

LEMMA 10. *Let $B \in \boldsymbol{Q}$ and $A$ be poor. Then*

$$B-A \neq \phi .$$

*Proof.* Let $A$ be a poor set. Then there are $D_0, D_1, \cdots, D_n, \cdots$ such that $\forall n<\omega[D_n \in \boldsymbol{D}_n]$ and $A \subset \bigcup_{n<\omega} D_n$.
By the induction on $n$, we define $g_n$ so that

$$g_n \in B^*-D_n^* .$$

In the case $n=0$, consider the set

$$E=\{g: \mathrm{dom}(g)=1 \wedge g \in B^*-D_0^*\} .$$

Then, by $D_0 \in \boldsymbol{D}_0$, we have $E \neq \phi$. Let $g_0$ be one of the functions in $E$. Let $g_n$ be already difined. Then we have

$$\overline{\overline{\{\nu : g_n * \nu \in B^*-D_n^*\}}}=\omega_\alpha .$$

Hence there is an ordinal $\nu_0$ such that $g_n * \nu_0 \in B^*-D_n^*$. Put $g_{n+1}=g_n * \nu_0$. Since $B$ is closed, $g^*=\bigcup_{n<\omega} g_n \in B$. On the other hand, by the definition of $g^*$, we obtain $g^* \notin \bigcup_{n<\omega} D_n$. Namely

$$g^* \in B-\bigcup_{n<\omega} D_n \subset B-A .$$

Namely, *if $A$ is closed, then $A$ is rich if and only if there is $B \in \boldsymbol{Q}$ such that $B \subset A$.*

LEMMA 11. *Let $A$ be a closed set, and $\psi$ be a function such that $\psi : A^* \longrightarrow \rho$. For $f : \omega \longrightarrow \rho$, put*

$$A^f=\{g \in A : \forall n<\omega[\psi(g \upharpoonright n)=f(n)]\}$$

*Then we have that $A^f$ is closed and*

$$A=\bigcup_{f \in \rho^\omega} A^f .$$

*Proof.* Let $f : \omega \longrightarrow \rho$, and let $g_0 \subset g_1 \subset \cdots \subset g_n \subset \cdots$ be a sequence of functions in $(A^f)^*$ such that $g^\#=\bigcup_{n<\omega} g_n \in \omega_\alpha^\omega$. Then we have

$$g^\# \upharpoonright n=h \upharpoonright n \text{ for some } h \in A^f .$$

Hence $\psi(g^\# \upharpoonright n)=\psi(h \upharpoonright n)=f(n)$. Namely $g^\# \in A^f$. This means that $A^f$ is closed. Now let $g \in A$. Define a function $f_1 : \omega \longrightarrow \rho$ by

$$f_1(n)=\psi(g \upharpoonright n) .$$

Then

$$g \in A^{f_1} \subset \bigcup_{f \in \rho^\omega} A^f .$$

Namely $A=\bigcup_{f \in \rho^\omega} A^f$.

LEMMA 12. *Let $A \in \boldsymbol{Q}$, $\psi : A^* \longrightarrow \rho$, $\overline{\overline{\rho^\omega}}<\omega_\alpha$ and $\omega_\alpha$ be a regular cardinal. Then for some $f : \omega \longrightarrow \rho$, $A^f$ is rich.*

*Proof.* Assume that for all $f: \omega \longrightarrow \rho$, $A^f$ is poor. Then there is $D_0^f, D_1^f, \cdots, D_n^f, \cdots$ such that $\forall n<\omega[D_n^f \in \boldsymbol{D}_n]$ and

$$A^f \subset \bigcup_{n<\omega} D_n^f .$$

By the fact that $A=\bigcup_{f \in \rho^\omega} A^f$, we have

$$A=\bigcup_{f \in \rho^\omega} A^f \subset \bigcup_{f \in \rho^\omega} \bigcup_{n<\omega} D_n^f=\bigcup_{n<\omega} \bigcup_{f \in \rho^\omega} D_n^f .$$

Since $\rho^\omega < \omega_\alpha$ and $\omega_\alpha$ is regular, we have

$$\bigcup_{f \in \rho^\omega} D_n^f \in \boldsymbol{D}^n .$$

Hence we conclude that $A$ is poor. This is a contradiction. Namely, for some $f: \omega \to \rho$, $A^f$ is rich.

LEMMA 13. *Let $B \in \boldsymbol{Q}$, $\overline{\overline{\rho^\omega}} < \omega_\alpha$ and $\omega_\alpha$ be a regular cardinal. Suppose that*

$$\forall g \in B^* \, \exists \nu < \rho [f_0 * g * B_g \in \boldsymbol{a}_{\mathrm{dom}(g)\,\nu}] ,$$

*where $B_g = \{h : g*h \in B\}$. Then there exist $f: \omega \to \rho$ and $A \in \boldsymbol{Q}$ such that $A \subset B$ and*

$$f_0 * A \in \boldsymbol{a}_{n\, f(n)} \quad \textit{for all } n < \omega .$$

*Proof*. We choose $\psi : B^* \to \rho$ so that

$$f_0 * g * B_g \in \boldsymbol{a}_{\mathrm{dom}(g)\, \psi(g)} .$$

Then by Lemma 12, there is a $f: \omega \to \rho$ such that $B^f$ is rich. Since $B^f$ is closed, there is a set $C \in \boldsymbol{Q}$ such that $B^f \supset C$. By the definition of $B^f$, for a fixed $n < \omega$, we have

$$f_0 * (g \restriction n) * B_{g \restriction n} \in \boldsymbol{a}_{n\, f(n)} \quad \text{for all } g \in B^f ,$$

*a fortiori*

$$f_0 * (g \restriction n) * C_{g \restriction n} \in \boldsymbol{a}_{n\, f(n)} \quad \text{for all } g \in C .$$

We claim that $V(f_0 * C) \subset \boldsymbol{a}_{n\, f(n)}$. Take any $f_0' * C'$ such that

$$f_0' * C' \subset f_0 * C .$$

Then there is a function $f_1' \in C^*$ such that $f_0' = f_0 * f_1'$. Let $g' \in C'$. Then

$$f_0 * ((f_1' * g') \restriction n) * C'_{(f_1' * g') \restriction n} \subset f_0' * C' .$$

On the other hand we have

$$f_0 * ((f_1' * g') \restriction n) * C'_{(f_1' * g') \restriction n} \subset f_0 * ((f_1' * g') \restriction n) * C_{(f_1' * g') \restriction n} \in \boldsymbol{a}_{n\, f(n)} .$$

Thus $V(f_0' * C') \cap \boldsymbol{a}_{n\, f(n)} \neq \phi$. Then, since $\boldsymbol{a}_{n\, f(n)}$ is regular open, $f * C \in \boldsymbol{a}_{n\, f(n)}$. By virture of the arbitrariness of $n$, we conclude that

$$f_0 * C \in \bigcap_{n < \omega} \boldsymbol{a}_{n\, f(n)} = \prod_{n < \omega} \boldsymbol{a}_{n\, f(n)} .$$

*Proof of Theorem* 3: Let $\omega_\alpha$ be a regular cardinal and $\forall \nu < \alpha [\overline{\overline{\omega_\nu^\omega}} < \omega_\alpha]$. Consider the complete Boolean algebra $\boldsymbol{B}^*$ defined above. Assume that

$$\prod_{n < \omega} \sum_{\nu < \omega_\beta} \boldsymbol{a}_{n\, \nu} = 1$$

for $\omega_\beta < \omega_\alpha$. Let $f_0 * A$ be any element of $\boldsymbol{P}$. Then by Lemma 7, 13, we have a set $f_0 * C \in \boldsymbol{P}$ and a function $f : \omega \to \omega_\beta$ such that $f_0 * C \subset f_0 * A$ and

$$f_0 * C \in \prod_{n<\omega} \boldsymbol{a}_{n\,f(n)} \,.$$

This means that $\bigcup_{f\in\omega_\beta^\omega} \prod_{n<\omega} \boldsymbol{a}_{n\,f(n)}$ is a dense set in $(\boldsymbol{P}, \leqq)$. Namely we obtain that

$$\sum_{f\in\omega_\beta^\omega} \prod_{n<\omega} \boldsymbol{a}_{n\,f(n)} = \boldsymbol{1} \,.$$

Combining this with Lemma 4, we obatin that

(1) $\forall\nu<\alpha[(\omega, \omega_\nu)$-DL holds in $\boldsymbol{B}^*[\omega_\alpha]]$,

(2) $(\omega, \omega_\alpha)$-DL fails to hold in $\boldsymbol{B}^*[\omega_\alpha]$.

Thus we complete the proof of Theorem 3.

LEMMA 14. *Let $cf(\omega_\alpha)=\omega_\beta>\omega$ and $\forall\nu<\alpha[\overline{\overline{\omega_\nu^\omega}}<\omega_\alpha]$. For every complete Boolean algebra $\boldsymbol{B}[\omega_\alpha]$, if*

*(1) $\forall\nu<\alpha[(\omega, \omega_\nu)$-DL holds in $\boldsymbol{B}[\omega_\alpha]]$.*

*(2) $(\omega, \omega_\beta)$-WDL ($(\omega, \omega_\beta)$-weak distributive law) holds in $\boldsymbol{B}[\omega_\alpha]$, then $(\omega, \omega_\alpha)$-DL holds in $\boldsymbol{B}[\omega_\alpha]$.*

*Proof.* Let $\prod_{n<\omega} \sum_{\nu<\omega_\alpha} \boldsymbol{a}_{n\nu} = \boldsymbol{1}$. Since $\omega_\beta = cf(\omega_\alpha)$, there is a function $t : \omega_\beta \to \omega_\alpha$ such that $\sup_{\nu<\omega_\beta} t(\nu) = \omega_\alpha$. By $(\omega, \omega_\beta)$-WDL, we have

$$\sum_{f\in\omega_\beta^\omega} \prod_{n<\omega} \sum_{\nu<t(f(n))} \boldsymbol{a}_{n\nu} = \boldsymbol{1} \,.$$

Since $\forall\nu<\alpha[(\omega, \omega_\nu)$-DL$]$ and $\omega<cf(\omega_\alpha)$, we have

$$\sum_{f\in\omega_\beta^\omega} \sum_{g\in\prod_{m<\omega} t(f(m))} \prod_{n<\omega} \boldsymbol{a}_{n\,g(n)} = \boldsymbol{1} \,.$$

This means that

$$\sum_{f\in\omega_\alpha^\omega} \prod_{n<\omega} \boldsymbol{a}_{nf(n)} = \boldsymbol{1} \,.$$

Namely $(\omega, \omega_\alpha)$-DL holds in $\boldsymbol{B}$.

*Notice.* Putting $\overline{\overline{\omega_\alpha^\omega}} = \omega_\beta$, by Hausdorff's theorem, we have $\omega_{\beta+1}^\omega = \omega_{\beta+1} \cdot \omega_\beta^\omega = \omega_{\beta+1}$. By this, there is a complete Boolean algebra satisfying

$$(\omega, \omega_\alpha^\omega)\text{-DL but not } (\omega, \omega_\alpha^{\omega+})\text{-DL}.$$

Assume GCH (generalized continuum hypotheses), then

$$\forall\nu<\alpha[(\omega, \omega_\nu)\text{-DL}] \to (\omega, \omega_\alpha)\text{-DL}$$

holds for all complete Boolean algebra if and only if $\alpha=\beta+1$ and $cf(\beta)=\omega$.

Next we consider the situation $2^\omega=\omega_1$. According to Kenneth Kunen,

$$(\omega, \omega)\text{-WDL} \to (\omega, \omega_1)\text{-WDL}$$

holds in any complete Boolean algebra. But by Theorem 3, we have a complete Boolean algebra $\boldsymbol{B}^*$ in which

$$(\omega, \omega_1)\text{-DL holds but not } (\omega, \omega_2)\text{-DL}.$$

By Lemma 14, we have $\neg(\omega, \omega_2)$-WDL in $\boldsymbol{B}^*$. This means that

$$(\omega, \omega_1)\text{-DL}, \quad \neg(\omega, \omega_2)\text{-WDL}$$

are satisfied in $\boldsymbol{B}^*$.

We consider the case where $\omega_\alpha$ is regular and $\forall \nu < \alpha[\overline{\overline{\omega_\nu^\omega}} < \omega_\alpha]$. Let the distance $d(f, g)$ for $f, g \in \omega_\alpha^\omega$ be defined by

$$d(f, g) = \begin{cases} 1/(n+1), & \text{where } n = \mu x[f(x) \neq g(x)], \\ 0, & \text{if } f = g. \end{cases}$$

Then $(\omega_\alpha^\omega, d)$ is a complete metric space. Let $\boldsymbol{P}'$ be the set of all $\omega_\alpha$-perfect sets and define order relation on $\boldsymbol{P}'$ by

$$A \geqq B \equiv A \subset B.$$

Then similarly as above we have that the complete Boolean algebra $\boldsymbol{B}'$ induced by this order structure satisfies the conditions (1), (2) in Theorem 3.

*Problem* 1. Assume that $\forall n < \omega[\overline{\overline{2^{\omega_n}}} < \omega_\omega]$. Let $(\boldsymbol{P}, \leqq)$ be the structure of $\prod_{n<\omega} \omega_n$-perfect sets with the relation of inverse inclusion. Let $\boldsymbol{B}_0$ be the complete Boolean algebra induced by this order structure.

$$\text{Does } (\omega_1, 2)\text{-DL hold in } \boldsymbol{B}_0\text{ ?}$$

More generaly

$$\text{Does } (\omega_n, 2)\text{-DL hold in } \boldsymbol{B}_0\text{ ?}$$

*Problem* 2. Assume that

$$\forall \nu < \alpha[\overline{\overline{\omega_\nu^{\omega_1}}} < \omega_\alpha] \text{ and } cf(\omega_\alpha) = \omega_\alpha \vee cf(\omega_\alpha) = \omega_1.$$

Is there a complete Boolean algebra $\boldsymbol{B}$ such that

(1) $\forall \nu < \alpha[(\omega_1, \omega_\nu)$-DL holds in $\boldsymbol{B}]$,

(2) $(\omega_1, \omega_\alpha)$-DL fails to hold in $\boldsymbol{B}$?

*Problem* 3. Assume that $\forall \nu < \alpha[\overline{\overline{\omega_\nu^\omega}} < \omega_\alpha]$ and $cf(\omega_\alpha) = \omega_\alpha$. Let

$$A_1 \cup A_2 = \omega_\alpha^\omega$$

Then is there a $\omega_\alpha$-perfect set $P$ such that

$$P \subset A_1 \vee P \subset A_2\text{ ?}$$

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Department of Mathematics
Tokyo University of Education
Otsuka Bunkyo-ku
Tokyo, Japan